[16th Workshop on Markov Processes and Related Topics]

A second order expansion in the local limit theorem for a symmetric irreducible branching random walk

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Branching Random Walks

A natural model that describe the evolution of a population of particles where spatial motion is present.

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Definition:

- Time 0, one initial particle $\varnothing$ located at 0;
- Time 1, $\varnothing$ is replaced by $N_\varnothing$ new particles i of generation 1, each with location $S_i = L_i, 1 \le i \le N$;
- Time n, each particle u of generation $n(|u| = n)$ is replaced by N_u new particles ui with location $S_{ui} = S_u + L_{ui}(1 \le N_u)$.

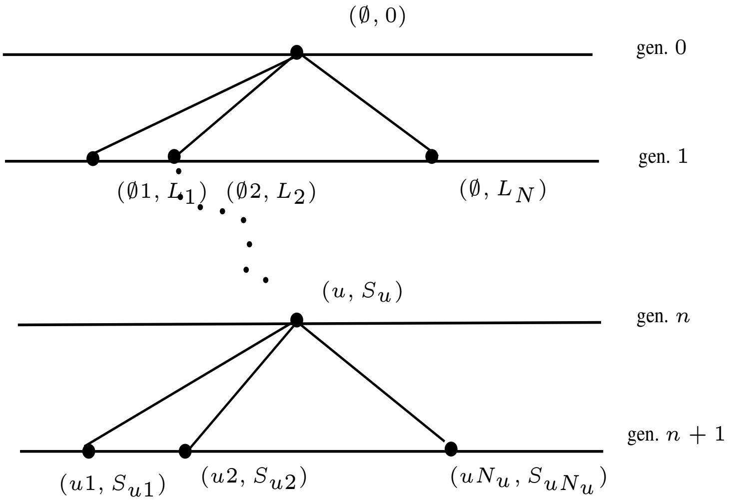
N_u: independent copies of N;

L_u: independent copis of L.

N and L random variables taking values in $\mathbb{N}$ and E resp.

Here E may be taken as $\mathbb{R}^d$ or $\mathbb{Z}^d$.

The case $d = 1$



$(\emptyset, 0)$
gen. 0
gen. 1
$(\emptyset 1, L_1)$
$(\emptyset 2, L_2)$
$(\emptyset, L_N)$
(u, S_u)
gen. n
gen. $n + 1$
$(u1, S_{u1})$
$(u2, S_{u2})$
(uN_u, S_{uN_u})

Question

Denote by $Z_n(\cdot)$ the counting measure of particles of generation n: for $B \subset \mathbb{R}$,

$$Z_n(B) = \sum_{u \in \mathbb{T}_n} \mathbf{1}_B(S_u).$$

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Question (Harris 1963)

Central limit theorems for appropriately normalised $Z_n(\cdot)$?

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Question (Harris 1963)

Central limit theorems for appropriately normalised $Z_n(\cdot)$?

T. E. Harris, *The theory of branching processes*, Die Grundlehren der Mathematischen Wissenschaften, Bd. 119, Springer-Verlag, Berlin, 1963.

History of CLT for BRW

Since Harris first proposed his conjecture on the question of central limit theorems for a branching random walk, the topic has been widely studied in various forms.

N. Kaplan and S. Asmussen, *Branching random walks. II*, Stochastic Processes Appl. **4** (1976), no. 1, 15–31.

Biggins J D. *The central limit theorem for the supercritical branching random walk, and related results*. Stoch Process Appl, **34**(1990): 255–274.

Overview of this Lecture

1. Asymptotic expansion in LLT for SBRW

2. Asymptotic expansion in LLT for symmetric irreducible BRW

3. Sketch of proofs

Simple branching random walk

The reproduction is governed by a supercritical Galton-Watson process and the migration of particles by a simple random walk in $\mathbb{Z}^d$.

All step size L_u connected with the particle u are independent copies of L with the law

$$\mathbb{P}(L = \mathbf{e}_j) = \mathbb{P}(L = -\mathbf{e}_j) = \frac{1}{2d}, \quad j = 1, 2, \cdots, d,$$

where $\mathbf{e}_j (1 \le j \le d)$ are the orthogonal unit vectors in $\mathbb{Z}^d$.

Notation

Define

$$m = \mathbb{E}N, \quad W_n = \frac{Z_n(\mathbb{Z}^d)}{m^n}.$$

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The following facts are well known(Athreya and Ney 1972):

- W_n is a martingale and there exists

$$W = \lim_{n\to\infty} W_n;$$

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$$W = \lim_{n\to\infty} W_n;$$

- If $m > 1, \mathbb{E}N \ln N < \infty$, then W is non-degenerate, $\mathbb{E}W = 1$ and $W > 0$ on the non-extinction set $\{Z_n \to \infty\}$.

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Denote by $\langle\cdot,\cdot\rangle$ the inner product in $\mathbb{R}^d$ and $\|\cdot\|$ the norm therein, i.e. for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$,

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_dy_d, \quad \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$

For the simple branching random walk,
Révész (1994) considered the local limit theorem and initiated the study of the convergence speed in local limit theorems .
Specially, Révész gave a conjecture on the exact rate of the convergence speed for the simple branching random walk, which was proved by Chen (2001, AAP).

- X. Chen, *Exact convergence rates for the distribution of particles in branching random walks*, Ann. Appl. Probab. **11** (2001), no. 4, 1242–1262.
- P. Révész, *Random walks of infinitely many particles*, World Scientific Publishing Co. Inc., River Edge, NJ, 1994.

Local limit theorem for SBRW

Revesz (1994)[P.76 Theorem 4.8] :

Theorem (LLT for d-SBRW)

Assume $\mathbb{P}(N \geq 1) = 1$, $m > 1$ *and* $\mathbb{E}N^2 < \infty$. *For each* $z = (z_1, z_2, \cdots, z_d) \in \mathbb{Z}^d$,

$$\frac{1}{2}\left(\frac{2\pi n}{d}\right)^{d/2} \frac{Z_n(z)}{m^n} \longrightarrow W \quad a.s., \tag{1}$$

as $n \to \infty$ *with* $n \equiv z_1 + z_2 + \cdots + z_d \pmod 2$.

P. Révész, *Random walks of infinitely many particles*, World Scientific Publishing Co. Inc., River Edge, NJ, 1994.

Two martingales

Révész gave a conjecture on the exact rate of the convergence speed for the simple branching random walk, and Chen demonstrated the strengthened version of the conjecture.

To this end, Chen introduced two important martingales:

$$N_{1,n} = \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} S_u \quad \text{and} \quad N_{2,n} = \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \left[\|S_u\|^2 - n \right]. \tag{2}$$

Under the condition $\mathbb{E}N^2 < \infty$, he proved their convergence. The limits $\mathcal{V}_1 \in \mathbb{R}^d$ and $\mathcal{V}_2 \in \mathbb{R}$ are given by

$$\mathcal{V}_j = \lim_{n \to \infty} N_{j,n} (j = 1, 2).$$

X. Chen, *Exact convergence rates for the distribution of particles in branching random walks*, Ann. Appl. Probab. **11** (2001), no. 4, 1242–1262.

Convergence rate in LLT for SBRW

Theorem (Chen 2001 AAP)

Suppose that $\mathbb{E}N^2 < \infty$. *Then for each* $x = (x_1, x_2, \cdots, x_d) \in \mathbb{Z}^d$, *as* $n \to \infty$ *provided that* $n \equiv x_1 + x_2 + \cdots + x_d \pmod 2$,

$$\frac{n}{d}\left[\frac{1}{2}\left(\frac{2\pi n}{d}\right)^{d/2}\frac{Z_n(x)}{m^n} - W\exp\left\{-\frac{d\|x\|^2}{2n}\right\}\right] \xrightarrow[a.s.]{} -\frac{1}{2}\mathcal{V}_2 + \langle x, \mathcal{V}_1\rangle.$$

A factor was missing in the rate function.

Generalization by G.

Theorem (G. 2017 SPA)

Assume $m > 1$ *and* $\mathbb{E}N(\ln N)^{1+\lambda} < \infty$ *for* $\lambda > 4(d+3)$. *Then for each* $x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$, *provided that* $n \equiv x_1 + x_2 + \cdots + x_d \pmod 2$,

$$\frac{n}{d}\left[\frac{1}{2}\left(\frac{2\pi n}{d}\right)^{d/2}\frac{Z_n(x)}{m^n} - W\exp\left\{-\frac{d\|x\|^2}{2n}\right\}\right]$$

$$\xrightarrow[a.s.]{n\longrightarrow\infty} -\frac{1}{2}\mathcal{V}_2 + \langle x, \mathcal{V}_1\rangle - \frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{W}.$$

Zhi-Qiang Gao, *Exact convergence rate of the local limit theorem for branching random walks on the integer lattice*, Stoch. Process. Appl. 127 (2017), no. 4, 1282 – 1296.

First order expansion in LLT

Theorem (G. 2017 SPA)

Assume $m > 1$ *and* $\mathbb{E}N(\ln N)^{1+\lambda} < \infty$ *for* $\lambda > 4(d+3)$. *Then for each* $z = (z_1, z_2, \dots, z_d) \in \mathbb{Z}^d$, *as* $n \to \infty$ *with* $n \equiv z_1 + z_2 + \cdots + z_d \pmod 2$, *a.s.*

$$\frac{1}{m^n} Z_n(z) = 2\Big(\frac{d}{2\pi n}\Big)^{d/2}\Big[W + \frac{d}{n}F_1(z)\Big] + o(\frac{1}{n^{1+d/2}}), \quad (3)$$

where

$$F_1(z) = \Big(-\frac{\|z\|^2}{2} - \frac{1}{4}\Big)W + \langle z, \mathcal{V}_1\rangle - \frac{1}{2}\mathcal{V}_2. \quad (4)$$

Question

Asymptotic expansions of $Z_n(z)$?

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Z.-Q. Gao, *A second order asymptotic expansion in the local limit theorem for a simple branching random walk in* $\mathbb{Z}^d$, Stoch. Process. Appl. **128** (2018), no. 12, 4000–4017.

Some martingales

Consider the following martingales

$$N_{4,n} = \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \left(\|S_u\|^4 - (2 + \frac{4}{d}) n \|S_u\|^2 + (1 + \frac{2}{d}) n^2 + \frac{2}{d} n \right);$$

$$N_{3,n} = \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \left(\|S_u\|^2 S_u - \left(1 + \frac{2}{d}\right) n S_u \right);$$

$$N_{2,n}^z = \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \left(\langle z, S_u \rangle^2 - \frac{n}{d} \|z\|^2 \right).$$

These martingales converge. The quantities $\mathcal{V}_2^z, \mathcal{V}_2, \mathcal{V}_4 \in \mathbb{R}$ and $\mathcal{V}_1, \mathcal{V}_3 \in \mathbb{R}^d$ are defined by

$$\mathcal{V}_j = \lim_{n \to \infty} N_{j,n} \quad j = 1, 2, 3, 4; \qquad \mathcal{V}_2^z = \lim_{n \to \infty} N_{2,n}^z.$$

Their convergence rates are important ingredients in the studies.

Second order expansion in LLT

Theorem (G. 2018 SPA)

Assume $m > 1$ *and* $\mathbb{E}N(\ln N)^{1+\lambda} < \infty$ *for some* $\lambda > 6(d+5)$. *Then for each* $z = (z_1, z_2, \ldots, z_d) \in \mathbb{Z}^d$, *as* $n \to \infty$ *with* $n \equiv z_1 + z_2 + \cdots + z_d \,(\mathrm{mod}\ 2)$,

$$\frac{1}{m^n} Z_n(z) = 2\Big(\frac{d}{2\pi n}\Big)^{d/2}\Big[W + \frac{d}{n}F_1(z) + \frac{d^2}{n^2}F_2(z)\Big] + o\Big(\frac{1}{n^{2+d/2}}\Big), \tag{5}$$

where $F_1(z)$ *is defined by* (8) *and*

$$F_2(z) = \Big(\frac{1}{8}\|z\|^4 + \frac{1}{8}\big(1 + \frac{4}{d}\big)\|z\|^2 - \frac{1}{32} + \frac{d}{48} + \frac{1}{24d}\Big)W - \Big(\frac{1}{4} + \frac{1}{d} + \frac{1}{2}\|z\|^2\Big)\langle z, \mathcal{V}_1\rangle + \Big(\frac{1}{4}\|z\|^2 + \frac{1}{8} + \frac{1}{2d}\Big)\mathcal{V}_2 + \frac{1}{2}\mathcal{V}_2^z - \frac{1}{2}\langle z, \mathcal{V}_3\rangle + \frac{1}{8}\mathcal{V}_4.$$

Overview of this Lecture

1. Asymptotic expansion in LLT for SBRW

2. Asymptotic expansion in LLT for symmetric irreducible BRW

3. Sketch of proofs

Symmetric irreducible BRW

We extend the result for SBRW to the case where the migration mechanism is governed by a finite range symmetric irreducible random walk on $\mathbb{Z}^d$.

A symmetric irreducible random walk

The step size L is a random variable with the probability law

$$\mathbb{P}(L = \mathbf{0}) = \zeta_0, \quad \mathbb{P}(L = \pm r\mathbf{e}_s) = \frac{1}{2}\zeta_{s,r}, \quad 1 \le s \le d, 1 \le r \le t_s, \qquad (6)$$

where each t_s $(1 \le s \le d)$ is a positive integer, $\zeta_0 \in [0,1)$, $\zeta_{s,r} \in [0,1)$, and

$$\zeta_{s,t_s} > 0, \qquad \zeta_0 + \sum_{s=1}^{d}\sum_{r=1}^{t_s}\zeta_{s,r} = 1.$$

Assume that the law (6) of L satisfies

$$\gcd\{r : \zeta_{r,s} > 0\} = 1, \quad s = 1, 2, \cdots, d, \tag{7}$$

where $\gcd$ denotes the greatest common divisor.
This assumption implies that

$$V = \{r\mathbf{e}_s : \zeta_{s,r} > 0, 1 \le r \le t_s, s = 1, 2, \cdots, d\}$$

is a *generating set* of $\mathbb{Z}^d$, which means that

$$\forall y \in \mathbb{Z}^d, \quad \exists \{k_{r,s}\} \subset \mathbb{Z}, \quad \text{s.t. } y = \sum_{r\mathbf{e}_s \in V} k_{r,s} r\mathbf{e}_s.$$

Moreover, the random walk S_n with such increment distribution L must be *irreducible* (meaning that each point in $\mathbb{Z}^d$ can be reached with positive probability Lawler& Limic(2010)).

G. F. Lawler and V. Limic, *Random walk: a modern introduction*, Cambridge Studies in Advanced Mathematics, vol. 123, Cambridge University Press, Cambridge, 2010.

Two cases

Under the hypothesis (7) for L, there are two possible cases:

(Ha) there exists one s such that either the set $\{r : \zeta_{s,r} > 0\}$ contains at least one odd integer and one even, or the set $\{r : \zeta_{s,r} > 0\}$ only contains odd integers and $\zeta_0 > 0$;

(Hb) $\zeta_0 = 0$ and for each $1 \le s \le d$, $\{r : \zeta_{s,r} > 0\}$ only contains odd integers.

Aperiodic case and Bipartite case

Denote by $\mathcal{A}_d$ the set of all probability distributions L with the law (6) satisfying (7) and (Ha).

When $L \in \mathcal{A}_d$, the random walk with increment distribution L is *aperiodic*, which means that each point on $\mathbb{Z}^d$ can be reached after n steps with positive probability for all n sufficiently large.

Denote by $\mathcal{B}_d$ the set of all probability distributions L with the law (6) satisfying (7) and (Hb).

For $L \in \mathcal{B}_d$, the associated random walk is *bipartite*, that means the random walks starting from a given point x_0 return to x_0 only after an even number of steps. In this case, $\mathbb{Z}^d$ is divided into two disjoint sets $\mathbb{Z}_o$ and $\mathbb{Z}_e$, such that the walk starting form the origin reaches the states set $\mathbb{Z}_o$ in an odd number of steps and reaches $\mathbb{Z}_e$ in an even number of steps.

Some notations

For $k \in \mathbb{N}$, set

$$\zeta_s(k) = \sum_{r=1}^{t_s} \zeta_{s,r} r^k, \quad 1 \le s \le d.$$

Denote by $\Gamma_k = \operatorname{diag}\big(\zeta_1(k), \cdots, \zeta_d(k)\big)$ the diagonal matrix with diagonal entries $\zeta_1(k), \cdots, \zeta_d(k)$.
It is easy to see that

$$\det \Gamma_k = \prod_{s=1}^{d} \zeta_s(k), \quad \operatorname{tr}(\Gamma_k^{-1}) = \sum_{r=1}^{d} \big(\zeta_r(k)\big)^{-1},$$

where $\det M$ is the determinant of a $d \times d$ matrix M, M^{-1} is its inverse, and $\operatorname{tr}(M)$ is its trace.

Some martingales

The random variables $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \in \mathbb{R}^d$ and $\mathcal{V}_2^z, \mathcal{V}_4 \in \mathbb{R}$ are defined by :

$$\mathcal{V}_j \overset{a.s.}{=\!=} \lim_{n\to\infty} N_{j,n} \quad j = 1,2,3,4; \quad \mathcal{V}_2^z \overset{a.s.}{=\!=} \lim_{n\to\infty} N_{2,n}^z,$$

$$N_{1,n} = \frac{1}{m^n} \sum_{u\in\mathbb{T}_n} S_u,$$

$$N_{2,n} = \frac{1}{m^n} \sum_{u\in\mathbb{T}_n} \Big(\langle S_u, \mathbf{e}_1\rangle^2 - n\zeta_1(2), \langle S_u, \mathbf{e}_2\rangle^2 - n\zeta_2(2), \cdots, \langle S_u, \mathbf{e}_d\rangle^2 - n\zeta_d(2)\Big),$$

$$N_{2,n}^z = \frac{1}{m^n} \sum_{u\in\mathbb{T}_n} \Big[\big\langle S_u, \Gamma_2^{-1} z\big\rangle^2 - n\left\langle \Gamma_2^{-1} z, z\right\rangle\Big],$$

$$N_{3,n} = \frac{1}{m^n} \sum_{u\in\mathbb{T}_n} \Big[\big\langle S_u, \Gamma_2^{-1} S_u\big\rangle S_u - (d+2)nS_u\Big],$$

$$N_{4,n} = \frac{1}{m^n} \sum_{u\in\mathbb{T}_n} \Big[\big\langle S_u, \Gamma_2^{-1} S_u\big\rangle^2 - (4+2d)n\left\langle S_u, \Gamma_2^{-1} S_u\right\rangle + d(d+2)(n^2+n) - \operatorname{tr}(\Gamma_4\Gamma_2^{-2})n\Big].$$

Theorem (G. 2021 MPRF)

Assume that $m > 1$, $N \geq 1$ *a.s.,* $\mathbb{E}N(\ln N)^{1+\lambda} < \infty$ *for some* $\lambda > 3(d+6)$ *and* L *obeys the law* (6)*. Then for each* $z = (z_1, z_2, \ldots, z_d) \in \mathbb{Z}^d$*, as* $n \to \infty$*,*

(I) in the case $L \in \mathcal{A}_d$*, a.s.*

$$\frac{1}{m^n} Z_n(z) = \frac{(2\pi n)^{-d/2}}{\sqrt{\det \Gamma_2}} \left[W + \frac{1}{n} F_1(z) + \frac{1}{n^2} F_2(z) \right] + \frac{1}{n^{2+d/2}} o(1),$$

(II) in the case $L \in \mathcal{B}_d$*, provided that* $n \equiv z_1 + z_2 + \cdots + z_d \pmod 2$*, a.s.*

$$\frac{1}{m^n} Z_n(z) = \frac{2(2\pi n)^{-d/2}}{\sqrt{\det \Gamma_2}} \left[W + \frac{1}{n} F_1(z) + \frac{1}{n^2} F_2(z) \right] + \frac{1}{n^{2+d/2}} o(1),$$

Gao Z.Q., *A second order expansion in the local limit theorem for a branching system of symmetric irreducible random walks*, Markov processes and related fileds,Markov Process. Related Fields, (27)2021, 439-466.

where

$$F_1(z) = \Big(\tau_d - \frac{1}{2}\left\langle z, \Gamma_2^{-1} z\right\rangle\Big)W + \left\langle \mathcal{V}_1, \Gamma_2^{-1} z\right\rangle - \frac{1}{2}\left\langle \mathcal{V}_2, \Gamma_2^{-1}\mathbf{1}\right\rangle, \tag{8}$$

$$\tau_d = \frac{1}{8}\mathrm{tr}(\Gamma_4\Gamma_2^{-2}) - \frac{1}{8}d(d+2), \tag{9}$$

$$\begin{aligned} F_2(z) = \Big(\frac{1}{8}\left\langle \Gamma_2^{-1}z, z\right\rangle^2 - \langle \Lambda_d z, z\rangle + \chi_d\Big)W + \Big\langle \mathcal{V}_1, \Big(2\Lambda_d - \frac{1}{2}\left\langle z, \Gamma_2^{-1}z\right\rangle \Gamma_2^{-1}\Big)z\Big\rangle \\ + \Big\langle \mathcal{V}_2, \Big(\frac{1}{4}\left\langle z, \Gamma_2^{-1}z\right\rangle \Gamma_2^{-1} - \Lambda_d\Big)\mathbf{1}\Big\rangle + \frac{1}{2}\mathcal{V}_2^z - \frac{1}{2}\left\langle \mathcal{V}_3, \Gamma_2^{-1}z\right\rangle + \frac{1}{8}\mathcal{V}_4, \end{aligned} \tag{10}$$

$$\Lambda_d = \frac{1}{16}\Big(\mathrm{tr}\Big(\Gamma_4\Gamma_2^{-2}\Big) - (d+2)(d+4)\Big)\Gamma_2^{-1} + \frac{1}{4}\Gamma_4\Gamma_2^{-3}, \tag{11}$$

$$\begin{aligned} \chi_d = -\frac{1}{64}(d+2)(d+4)\mathrm{tr}\Big(\Gamma_4\Gamma_2^{-2}\Big) + \frac{1}{12}\mathrm{tr}\Big(\Gamma_4^2\Gamma_2^{-4}\Big) + \frac{1}{128}\Big(\mathrm{tr}\Big(\Gamma_4\Gamma_2^{-2}\Big)\Big)^2 \\ - \frac{1}{48}\mathrm{tr}\Big(\Gamma_6\Gamma_2^{-3}\Big) + \frac{1}{384}d(d+2)(d+4)(3d+2). \end{aligned} \tag{12}$$

Overview of this Lecture

1. Asymptotic expansion in LLT for SBRW

2. Asymptotic expansion in LLT for symmetric irreducible BRW

3. Sketch of proofs

Ideas of the proofs

- The key decomposition
- Borel-Cantelli Lemma,Truncation methods, Moment Inequalities for sums of independent random variables.
- A second order expansions in the local limit theorem for a finite range symmetric random walk.

From the additivity property of the branching process, it follows that

$$Z_n(z) = \sum_{u \in \mathbb{T}_{k_n}} \sum_{v \in \mathbb{T}_{n-k_n}(u)} \mathbf{1}_{\{S_{uv}=z\}}, \tag{13}$$

and

$$\mathbb{E}_{\mathscr{D}_{k_n}} \left(\sum_{v \in \mathbb{T}_{n-k_n}(u)} \mathbf{1}_{\{S_{uv}=z\}} \right) = m^{n-k_n} \mathbb{P}\Big(\widetilde{S}_{n-k_n} = z - y \Big)\Big|_{y=S_u}.$$

Then we have the following decomposition:

$$\begin{aligned} \frac{Z_n(z)}{m^n} &= \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left(\frac{\sum_{v \in \mathbb{T}_{n-k_n}(u)} \mathbf{1}_{\{S_{uv}=z\}}}{m^{n-k_n}} - \mathbb{P}(\widetilde{S}_{n-k_n} = z - y)\Big|_{y=S_u} \right) \\ &\quad + \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}(\widetilde{S}_{n-k_n} = z - y)\Big|_{y=S_u} =: \mathbb{D}_{1,n} + \mathbb{D}_{2,n}. \end{aligned} \tag{14}$$

Lemma 1

$$n^{2+\frac{d}{2}}\mathbb{D}_{1,n} \xrightarrow{n\to\infty} 0 \quad \text{a.s.} \tag{15}$$

Lemma 2 *As $n \to \infty$, a.s.*

(I) when $L \in \mathcal{A}_d$,

$$\mathbb{D}_{2,n} = \frac{(2\pi n)^{-d/2}}{\sqrt{\det \Gamma_2}}\left[W + \frac{1}{n}F_1(z) + \frac{1}{n^2}F_2(z)\right] + \frac{1}{n^{2+d/2}}o(1); \tag{16}$$

(II) when $L \in \mathcal{B}_d$, provided $n \equiv z_1 + z_2 + \cdots + z_d \pmod 2$,

$$\mathbb{D}_{2,n} = \frac{2(2\pi n)^{-d/2}}{\sqrt{\det \Gamma_2}}\left[W + \frac{1}{n}F_1(z) + \frac{1}{n^2}F_2(z)\right] + \frac{1}{n^{2+d/2}}o(1), \tag{17}$$

where $F_1(z)$ and $F_2(z)$ are defined by (8) *and* (10) *respectively.*

The developed methods can be used to obtain asymptotic expansions of orders 3, 4, 5, etc., but we have not yet found a simple and unified method.

Thank you!

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Reference

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Thank you for your attention!